Wavelets and Edge Detection
CS698 Final Project

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INTRODUCTION

Wavelets have had a relatively short and troubled history. They seem to be forever confined to footnotes in textbooks on Fourier theory. It seems that there is little that can be done with wavelets that cannot be done with traditional Fourier analysis.

Stephane Mallat was not the father of wavelet theory, but he is certainly an evangelist. His textbook on the subject, A Wavelet Tour of Signal Processing [1], contains proofs about the theory of wavelets, and a summation about what is known about them with applications to signal processing. One of his many papers, *Characterization of Signals from Multiscale Edges [2]*, is frequently cited as a link between wavelets and edge detection. Mallat's method not only finds edges, but classifies them into different types as well. Mallat goes on to describe a method of recovering complete images using only the edges, but we will not implement it in this project. In this project, we study this paper, and implement the method of Mallat to multiscale edge detection and analysis.

We will first present a short background on wavelet theory. Then we will describe the different types of edges that exist in images, and how they can be characterized using a Lipschitz constant. Next, we describe the algorithm for the wavelet transform, from the Mallat paper. Finally, we show the results of applying the algorithm to a test image, and a real image.

WAVELET ANALYSIS

THEORY

It is best to describe wavelets by showing how they differ from Fourier methods. A signal in the time domain is described by a function $f(t)$, where t is usually a moment in time. When we apply the Fourier transform to the signal, we obtain a function $F(\omega)$ that takes as input a frequency, and outputs a complex number describing the strength of that frequency in the original signal. The real part is the strength of the cosine of that frequency, and the imaginary part is the strength of the sine.

One way to obtain the Fourier transform of a signal is to repeatedly correlate the sine and cosine wave with the signal. When the results high valued, the coefficients of the Fourier transform will be high. Where the signal or the wave is close to zero, the coefficients will be low.

Fourier analysis has a big problem, however. The sine and cosine functions are defined from $-\infty$ to ∞. The effects of each frequency are analyzed as if they were spread over the entire signal. For most signals, this is not the case. Consider music, which is continuously varying in pitch. Fourier analysis done on the entire song tells you which frequencies exist, but not where they are.

The short time Fourier transform (STFT) is often used when the frequencies of the signal vary greatly with time. [3] In the JPEG image encoding standard, for example, the image is first broken up into small windows with similar characteristics. The Fourier transform is not applied to the entire image, but only to these small blocks. The disadvantage of this technique can be seen at high compression ratios, when the outlines of the blocks are clearly visible artifacts.

A second disadvantage is in resolution of analysis. When larger windows are used, lower frequencies can be detected, but their position in time is less certain. With a smaller window, the position can be determined with greater accuracy, but lower frequencies will not be detected.

The wavelet transform helps solve this problem. Once applied to a function $f(t)$, it provides a set of functions $W_s f(t)$. Each function describes the strength of a *wavelet* scaled by factor s at time t. The wavelet extends for only a short period, so its effects are limited to the area immediately surrounding t. The wavelet transform will give information about the strengths of the frequencies of a signal at time t.

In the first pages of his treatise [1], Mallat defines a wavelet as a function of zero average,

$$\int_{-\infty}^{\infty} \psi(t)dt = 0$$

which is dilated with scale parameter s, and translated by u:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

Unlike the sine and cosine functions, wavelets move toward quickly zero as their limits approach to +/-∞.

In [2], Mallat notes that the derivative of a smoothing function is a wavelet with good properties. Such a wavelet is shown in Figure 1.

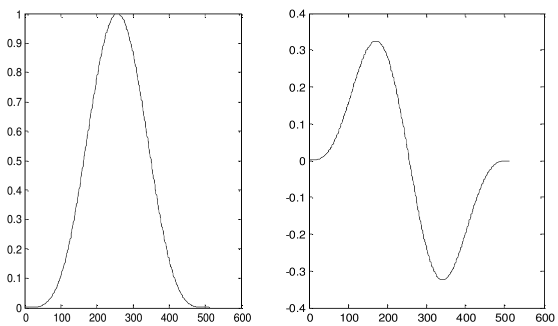


Figure 1: A smoothing function, and its corresponding wavelet.

By correlating the signal with this function at all possible translations and scales, we obtain the continuous wavelet transform.

The transformation also increases the dimension of the function by one. Since we have both a scaling and position parameter, a 1-D signal will have a 2-D wavelet transform. As an illustration, in Figure 2 we show the wavelet transform of a single scan line of an image, calculated using the algorithm in [2] (See Appendix A). The frequencies decrease from top to bottom, and pixel position increases from left to right. The edges in the signal result in funnel-shaped patterns in the wavelet transform.

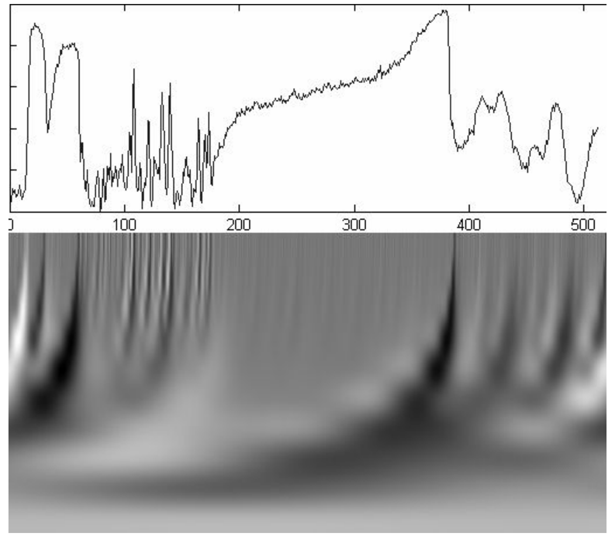
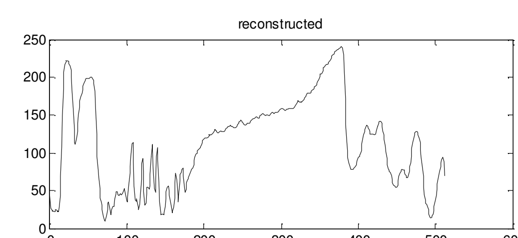


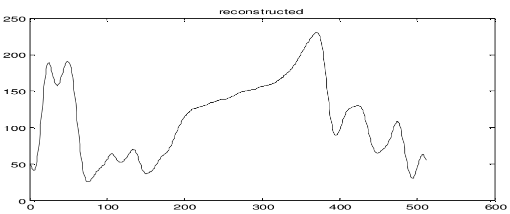
Figure 2: The 512th scanline of the famous Lena image, and its wavelet transform. Pixel position increases from left to right, and frequency increases from bottom to top. Only nine scales were used, but they are stretched to simulate a continuous transform, which is more illustrative.

Like the Fourier transform, the wavelet transform is invertible. However, it is easier to throw away information based on position. In the Fourier domain, if you were to try to eliminate noise by simply throwing away all of the information in a certain frequency band, you would get back a smoothed signal, with rounded corners, because all frequencies contribute to larger structures in all parts of the signal. With the wavelet transform, however, it is possible to selectively throw out high frequencies in areas where they do not contribute to larger structures. Indeed, this is the idea behind wavelet compression.

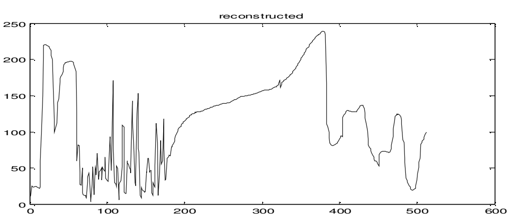
Here is the scan line from the Lena image, with the highest frequency wavelet coefficients removed:



The signal keeps the same structure as the original, but is smoother. Here is the same signal with the three highest dyadic[1] frequency bands removed:



The signal is smoother, but the edges are rounder. So far, this frequency removal is equivalent to smoothing the signal with a Gaussian. The true power of the wavelet transform is revealed, however, when we selectively remove wavelet coefficients from the first three dyadic frequency bands only in positions where they are weak (in this case, less than +/-20):



Here, the signal retains much of its original character. Most edges remain sharp. This simple algorithm for noise removal could be improved further if it did not change the weak coefficients in areas where they contribute to the larger structure. To do this, one would need to consider the coefficients across all scales, and determine the positions of the edges of the signal. In his paper, Mallat presents a way to do just that.

WAVELET TRANSFORM TYPES

There are numerous types of wavelet transforms. The first is the continuous wavelet transform (CWT). Dispite its name, the wavelet transform can be calculated on discrete data. All possible scaling factors are used, starting at 1 and increasing to the number of samples in the signal. However, the CWT is computationally expensive, and for most applications, a dyadic method is used instead.

In the dyadic wavelet transform, we use only scales that are powers of two. With the careful choice of an appropriate wavelet, this covers the entire frequency domain. At the scale *s=1*, the image is smoothed by convolving it with a smoothing function. At scale *s=2*, the smoothing function is stretched, and the image is convolved with it again. The process is repeated for *s=4, s=8*, etc., until the smoothing function is as large as the image. At each level, the wavelet transform contains information for every position *t* in the image. This method is used by Mallat.

Most applications today, however, use an even more optimal method. Since the image is smoothed at each step by a filter, the image only contains half of the frequency information, and needs half as many samples. So the number of samples in the image is reduced at each stage as well. As a result, the wavelet transform is reduced to the same size as the original signal. Mallat avoids this optimization because he needs the redundant information to recover the image using only its modulus maxima (edges).

CHARACTERIZATION OF EDGES

When the wavelet transform is used with a smoothing function, it is equivalent to Canny edge detection [4]. The derivative of a Gaussian is convolved with the image, so that local maxima and minima of the image correspond to edges. Note Figure 2, in which large drops are characterized by black funnels, and leaps result in white funnels. It is clear that by examining the wavelet transform, we can extract a lot of information about the edges. For example, we can see whether it is a gradual change or a leap, or whether it is a giant cliff, or a momentary spike, by looking only at the wavelet representation.

Edges are characterized mathematically by their Lipschitz regularity. We say that a function is uniformly Lipschitz α, over the interval (a,b), if, and only if, for every x, x_0 in the interval, there is some constant K such that:

$$\left|f(x)-f(x_0)\right| \leq K\left|x-y\right|^{\alpha}$$

---

[1] Dyadic: based on successive powers of two

The area over the interval will never have a slope that is steeper than the constant K. [5]

Mallat shows that the Lipschitz continuity is related to the wavelet transform, and that if the wavelet transform is Lipschitz α, the function is also Lipschitz α:

$$\left|W_{2^j} f(x)\right| \le K(2^j)^{\alpha}$$

The conclusions are summarized in the following table.

α constraint	**Meaning**	**Impact on Wavelet transform**
0 < α <= 1	f(x) is differentiable at x0. The change is gradual.	Amplitude decreases with scale.
α = 0	F(x) is not differentiable at x0. A sharp change or discontinuity exists.	Amplitude remains the same across scales.
-1 <= α < 0	F(x) is an impulse at x0.	Amplitude decreases with scale.

In Figure 3, an artificial 1-D signal is shown to illustrate the effects of a variety of edge types on the dyadic wavelet transform.

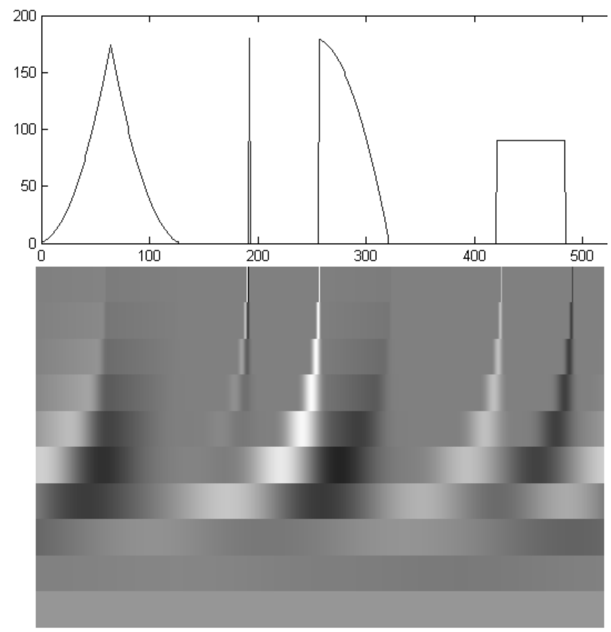


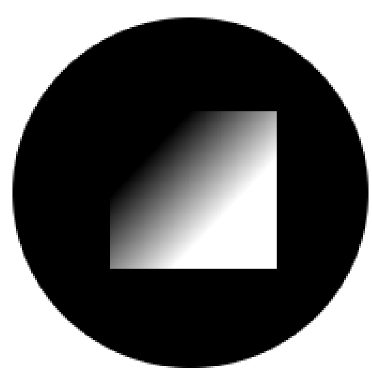
Figure 3

The algorithm can be extended to 2 dimensions. We say that a 2-D image is Lipschitz α in the box (x0, y0) – (x1, y2) if and only if there is a constant K, such that for any two points in the box, [2]

$$\left|f(x,y) - f(x_0,y_0)\right| \le K(\left|x - x_0\right|^2 + \left|y - y_0\right|^2)^{\alpha/2}$$

EDGE DETECTION ALGORITHM

The algorithm for performing the edge detection is as follows. In the following examples, this source image will be used.



When the wavelet transform of the image is performed, it results in two stacks of images. Since the image is 256x256 pixels, eight scaling levels are used, and each stack contains eight images. One image stack contains the separable horizontal filtering, and the other contains the vertical filtering. In the pictures that follow, the scaling factor *s=4* is shown. However, the algorithm is performed at all dyadic scaling levels.

At each step, the image is convolved with a wavelet to obtain the coefficients at that level. It is then smoothed with a Gaussian of increasing scale. Both the wavelet and Gaussian filtering is done using separate 1-D filters vertically and horizontally.

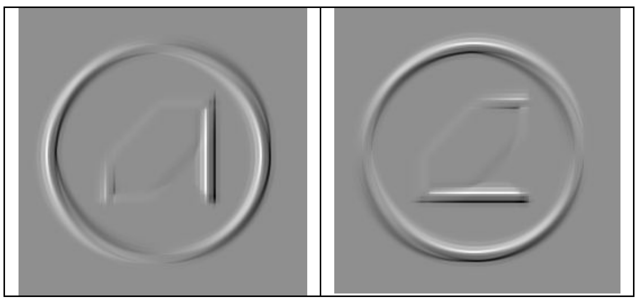


Figure 4: $W_s^1 f(x, y)$ **and** $W_s^2 f(x, y)$

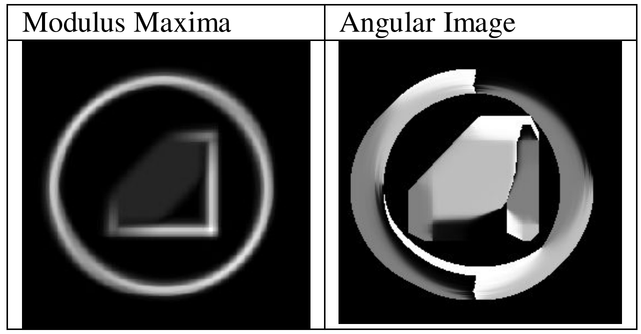
The modulus maxima image combines the two filtered images, and it is calculated using the formula:

$$M_s f(x, y) = \sqrt{\left|W_s^1 f(x, y)^2\right| + \left|W_s^2 f(x, y)\right|^2}$$

The angular image is calculated using

$$\arctan\left(\frac{W_s^2 f(x, y)}{W_s^1 f(x, y)}\right)$$

The result is shown below. The horizontal and vertical images form a gradiant image. The modulus maxima image is the scalar value of the vector at each point, and the angular image is the angle. In the angular image, low values represent 0 degrees from the horizontal, and higher values represent 90 degrees.



Finally, the lines of the maxima are found, using the information from both the modulus and angular image. Curiously, Mallat does not use a 2nd derivative to find them. Instead, he proposes a simple algorithm. A five point discrete derivative function was tried for this, but it did not perform better than Mallat's simple algorithm. A pixel is a modulus maximum if it is larger than its two neighbours long the angle of the gradient vector. A pixel has only eight neighbours, however. In my implementation of the algorithm, the angles from 0 to 2π are divided into 45° sections as illustrated in the figure below, so that two of the eight neighbouring pixels can be chosen to be compared to the centre pixel.

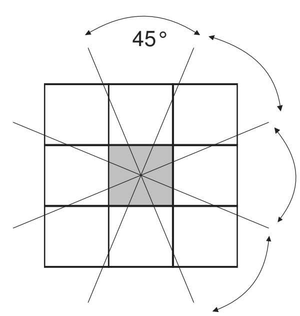
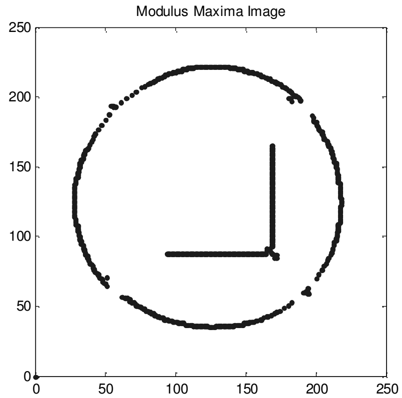


Figure 5: Angles are divided into sections to choose the maxima among neighbouring pixels.

To help detect only salient features, the maxima with a value above a certain threshold are taken and plotted.



The edge points can then be gathered together into "chains". For point that is a maxima, we can join it with the point closest to it if they have similar angles. They thus define the multiscale image edges.

The results of the modulus maxima edge detection on a real image are shown in Figure 6. For this project, we have selected an image of sharp railway tracks, shown through a blurred railing in the foreground.

At the finest scaling factor, the edges around gravel on the train tracks show up, but the blurred round ring does not. Being a larger feature, it does not appear until level 3.

The sharpest edges of all are the tracks themselves. They appear in all levels of the transform.

CONCLUSION

In this project, we have presented the main ideas of wavelet theory. Like the Fourier transform, wavelets give the strength of frequencies in a signal. Unlike the Fourier transform, they give the strength of the frequency at a certain moment in time. This property can be exploited as a method of multiscale edge detection.

Edges can be classified into different types, and they are characterized by their Lipschitz continuity. This continuity can be derived by observing the evolution of the wavelet transform across multiple scales. This can be seen by eye in the 1-D case. For the 2-D case, edge detection was implemented at multiple scales, and the algorithm of

REFLECTIONS

Mallat's algorithm for edge detection using wavelets is like Canny edge detection, but he claims to be able to characterize the edges by studying the evolution of the wavelet transform across multiple scales, and thus deriving the Lipschitz value associated with the edges. The algorithm seems to work well for one dimensional signals. However, it seems to break down for two dimensional signals. Mallat leaves many unanswered questions in his paper. The algorithm to chain maxima together is vague, and depends on many tunable factors to get right. For example, how close should two points be to be considered part of a chain? However, the chains must also be associated across image scales. It unclear how to disambiguate chains that lie close together.

Making the image periodic in order to convolve it for the wavelet transform results in distortions at the lower frequency levels, because the convolutions begin to wrap around from one side of the image to the other. At the lowest frequency scales, the wavelet maxima are unusable. It would have been better to use a different method of extending the image, such as simply placing it in a bed of zeros, and discounting the resulting edge from the results.

As a method of multiscale edge detection, wavelet analysis works on some levels but falls short on others. It works because it finds the edges and their angles, but in this regard it is very similar to doing Canny edge detection on multiple levels of scale. As shown in the 1-D case, it is possible to glean more information about the type of edge from the evolution across scales, but difficult to accomplish in the 2-D case.

WAVELET TRANSFORM AND EDGE DETECTION OF AN IMAGE

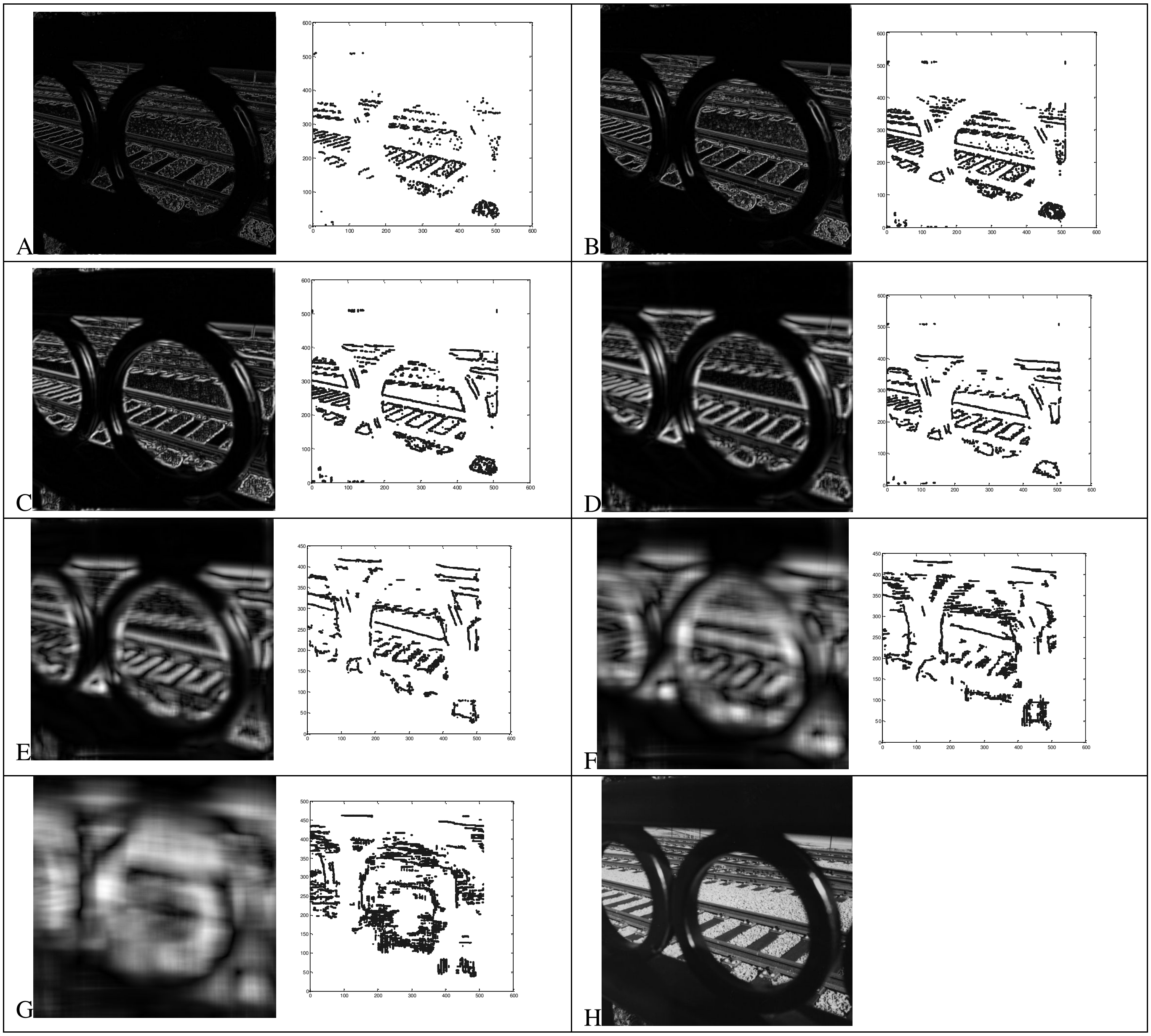


Figure 6: The wavelet transform and exact modulus maximus detection applied to a test image. H is the original image, and A-G are the modulus maxima at increasing levels of scaling factors. The last image, G, is distorted due to wrapping of the image for convolution.

Appendix A – Matlab Source Code

The dwt function calculates the dyadic wavelet transform of a 1-D function, using the algorithm described in [2]. The wavelets used are the ones described in the same paper, and pictured in Figure 1. In all of this code, the Matlab *mod* function is used to make the input signal appear to be periodic, for the purposes of convolution. For example, if you wanted to extract the12th element of an eight pixel scan line, mod(12-1, 8)+1 would calculate the correct pixel index to be 4.

```
function [W] = dwt( F )

SizeF = size(F);
N = SizeF(1);
% N should be a power of two.

J = log(N)/log(2);

W = zeros(N, J+1);

% Prepare normalization coefficients
LambdaTable = [ 1.50 1.12 1.03 1.01 ];
figure;
plot(F);
title('Original Signal');
j = 0;
while j < J
    p = 2 ^ j - 1;

    % Which normalization coefficient to use?
    Lambda = 1;
    if j < 4
        Lambda = LambdaTable(j+1);
    end

    % convolve the function with G, as if G has 2^j-1 zeros in between the
    % coefficients.
    for i = 1:N
        W(i,j+1) = (-2 * F(i) + 2 * F ( mod( i + p, N ) + 1 )) / Lambda;
    end

    %figure;
    %plot(W(:,j+1));

    % To get the next version of F, convolve it with H,, if H has w^j - 1
    % zeros between the coefficients.
    S = zeros(N,1);
    for i = 1:N
        S(i) = 0.125 * F( mod( i - p - 2, N) + 1 ) + ...
               0.375 * F(i) + ...
               0.375 * F( mod( i + p, N ) + 1 ) + ...
               0.125 * F( mod( i + p * 2 + 1, N ) + 1 );
    end

    F = S;

    j = j + 1;
end

W(:,J+1) = S;
```

The 1-D inverse wavelet transform is also implemented:

```
function [S] = idwt( W )

[N, J] = size(W);
S = W(:,J);
J = J - 1;

j = J;


% Prepare normalization coefficients
LambdaTable = [ 1.50 1.12 1.03 1.01 ]';

while j > 0
    p = 2 ^ (j-1) - 1; % Number of zeros between H, G, K coefficients
```

```
    % Which normalization coefficient to use?
    Lambda = 1;
    if j < 4
        Lambda = LambdaTable(j);
    end

    % Calculate the K part
    K = zeros(N, 1);
    for i = 1:N
        K(i) = 0.0078125 * W( mod( i - 3*p - 4, N) + 1, j ) + ...
               0.054685  * W( mod( i - 2*p - 3, N) + 1, j ) + ...
               0.171875  * W( mod( i - p - 2, N) + 1, j ) + ...
               -0.171875 * W(                      i, j ) + ...
               -0.054685 * W( mod( i + p    , N) + 1, j ) + ...
              -0.0078125 * W( mod( i + p * 2 + 1 , N) + 1, j );
    end

    % Calculate the ~H part.
    H = zeros(N, 1);
    for i = 1:N
        H(i) = 0.125 * S(mod( i - 2*p - 3, N ) + 1 ) + ...
               0.375 * S(mod( i - p - 2, N ) + 1 ) + ...
               0.375 * S(i) + ...
               0.125 * S(mod( i +p, N ) + 1 );
    end

    S = K * Lambda + H;

    j = j - 1;
end
```

The dwt2 function calculates the dyadic wavelet transform of a two dimensional image. It returns a three dimensional matrix W, which is a stack images representing the wavelet transform at dyadic scales.

```
function [Wx, Wy] = dwt2( F )

SizeF = size(F);
N = SizeF(1);

J = log(N)/log(2);

Wx = zeros(N, N, J+1);
Wy = zeros(N, N, J+1);


LambdaTable = [ 1.50 1.12 1.03 1.01 ];
figure;
imshow( F );
title('Original Signal');

S = zeros(N,N);

figure;

j = 0;
while j < J

    p = 2 ^ j - 1;

    % Which normalization coefficient to use?
    Lambda = 1;
    if j < 4
        Lambda = LambdaTable(j+1);
    end


    for y = 1:N
        for x = 1:N
            Wx(y,x,j+1) = (-2 * F(y,x) + ...
                          2 * F(y,mod(x+p,N)+1) ) / Lambda;

            Wy(y,x,j+1) = (-2 * F(y,x) + ...
                    2 * F(mod(y+p,N)+1,x) ) / Lambda;

            S(y,x) = 0.125 * F( y, mod( x - p - 2, N) + 1 ) + ...
                   0.375 * F(y, x) + ...
```

```
                   0.375 * F(y,  mod( x + p, N ) + 1 ) + ...
                   0.125 * F(y, mod( x + p * 2 + 1, N ) + 1 ) + ...
                   0.125 * F( mod( y - p - 2, N) + 1, x ) + ...
                   0.375 * F(y, x) + ...
                   0.375 * F(mod( y + p, N ) + 1,x ) + ...
                   0.125 * F( mod( y + p * 2 + 1, N ) + 1,x );

         end

     end

     subplot(J,2,j*2+1);
     imshow(Wx(:,:,j+1), [min(min(Wx(:,:,j+1))) max(max(Wx(:,:,j+1)))]);
     subplot(J,2,j*2+2);
     imshow(Wy(:,:,j+1), [min(min(Wy(:,:,j+1))) max(max(Wy(:,:,j+1)))]);

     F = S;
     j = j + 1;
end

%W(:,:,J+1) = S;
figure;
imshow(S, 256);
```

Here, we use the inverse dwt on the 1-d scanline of the lena image, to remove noise below a certain threshold, but retain the sharp features where strong coefficients exist.

```
I = double(imread('lena.png'));
Size = 512;
M = I(512,:)';

W = dwt(M);

for j = 1:3
    for i = 1:512
        if abs(W(i,j)) < 20
            W(i,j) = 0;
        end
    end
end

F = idwt(W);
figure;
plot(F);
title('reconstructed');

figure;
imshow( W', [ min(min(W)) max(max(W)) ] );
```

REFERENCES

1 S. Mallat, *A Wavelet Tour of Signal Processing*, London: Academic Press, 1999

2 S. Mallat, "Characterization of Signals from Multiscale Edges," *IEEE. Trans. Patt. Anal. Machine Intell.,* vol. PAMI-14, pp. 710-732, 1992.

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5 Wikipedia. "Lipschitz Continuity," http://en.wikipedia.org/wiki/Lipschitz_continuity